10

CHAPTER

# Fourier – Series Half Range Fourier Sine and Cosine Series

## 10.1 INTRODUCTION

Fourier series is an expansion of a periodic (Simple oscillating) function $f(x)$ in terms of an infinite sum of Sines and Cosines. Fourier series also make use of the orthogonality relationships of the sine and cosine functions. Half range Fourier series is a Fourier series defined on an interval instead of the more common, with the implication that the analyzed function should be extended to as either an even ($f(-x) = f(x)$) or odd function ($f(-x) = -f(x)$). This allows the expansion of the function in a series solely of Sines or Cosines.

## 10.2 FOURIER SERIES

Here we will express a non-sinusoidal periodic function into a fundamental and its harmonics. A series of sines and cosines of an angle and its multiples of the form:

$$\frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

is called the Fourier series, where $a_0, a_1, a_2, \ldots a_n, \ldots b_1, b_2, b_3 \ldots b_n \ldots$ are constants.

A periodic function $f(x)$ can be expanded in a Fourier Series. The series consists of the following:

(*i*) A constants term $a_0$ (called d.c. component in electrical work).

(*ii*) A component at the fundamental frequency determined by the values of $a_1, b_1$.

(*iii*) Components of the harmonics (multiples of the fundamental frequency) determined by $a_2, a_3 \ldots b_2, b_3 \ldots$. And $a_0, a_1, a_2 \ldots, b_1, b_2 \ldots$ are known as *Fourier coefficients* or Fourier constants.

**Note.** (1) When the function and its derivatives are continuous then the function can be expanded in powers of $x$ by Maclaurin's theorem.

(2) But by Fourier series we can expand continuous and discontinuous both types of functions under certain conditions.

## 10.3 PERIODIC FUNCTIONS

If the value of each ordinate $f(t)$ repeats itself at equal intervals in the abscissa, then $f(t)$ is said to be a periodic function.

If $f(t) = f(t + T) = f(t + 2T) = \ldots$ then $T$ is called the period of the function $f(t)$.



**For example:**

The period of $\sin x$, $\cos x$, $\sec x$, and $\operatorname{cosec} x$ is $2\pi$.

The period of $\tan x$ and $\cot x$ is $\pi$.

$\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \ldots$ so $\sin x$ is a periodic function with the period $2\pi$.

$$\sin 5x = \sin(5x + 2\pi) = \sin 5\left(x + \frac{2\pi}{5}\right), \text{ Period} = \frac{2\pi}{5}$$

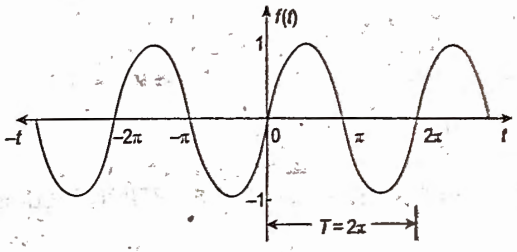
$$\cos 3x = \cos(3x + 2\pi) = \cos 3\left(x + \frac{2\pi}{3}\right), \text{ Period} = \frac{2\pi}{3}$$

$$\cos\frac{2n\pi x}{k} = \cos\left(\frac{2n\pi x}{k} + 2\pi\right) = \cos\frac{2n\pi}{k}\left(x + \frac{2\pi k}{2n\pi}\right)$$

$$= \cos\frac{2n\pi}{k}\left(x + \frac{k}{n}\right), \text{ Period} = \frac{k}{n}$$

$$\tan 2x = \tan(2x + \pi) = \tan 2\left(x + \frac{\pi}{2}\right), \frac{\pi}{2}$$

This is also called sinusoidal periodic function.




## 10.4 DIRICHLET'S CONDITIONS FOR A FOURIER SERIES

If the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single-valued

(2) is bounded

(3) has at most a finite number of maxima and minima

(4) has only a finite number of discontinuous

(5) is $f(x + 2\pi) = f(x)$ for values of $x$ outside $[-\pi, \pi]$, then

$$S_P(x) = \frac{a_0}{2} + \sum_{n=1}^{P} a_n \cos nx + \sum_{n=1} b_n \sin nx$$

Converges to $f(x)$ as $P \to \infty$ at values of $x$ for which $f(x)$ is continuous and the sum of the series is equal to $\frac{1}{2}[f(x+0) + f(x-0)]$ at points of discontinuity.

## 10.5 ADVANTAGES OF FOURIER SERIES

1. Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous function do not exist. (This is not true for Taylor's series).
2. The Fourier series is useful in expanding the periodic functions since outside the closed intervals, there exists a periodic extension of the function.

3. Expansion of an oscillating function by Fourier series gives all modes of oscillation (fundamental and all overtones) which is extremely useful in physics.
4. Fourier series of a discontinuous function is not uniformly convergent at all points.
5. Term by term integration of a convergent Fourier series is always valid, and it may be valid if the series is not convergent. However, term by term, differentiation of a Fourier series is not valid in most cases.

## 10.6 USEFUL INTEGRALS

The following integrals are useful in Fourier Series.

(i) $\int_0^{2\pi} \sin nx \, dx = 0$ (ii) $\int_0^{2\pi} \cos nx \, dx = 0$

(iii) $\int_0^{2\pi} \sin^2 nx \, dx = \pi$ (iv) $\int_0^{2\pi} \cos^2 nx \, dx = \pi$

(v) $\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$ (vi) $\int_0^{2\pi} \cos nx \cos mx \, dx = 0$

(vii) $\int_0^{2\pi} \sin nx \cdot \cos mx \, dx = 0$ (viii) $\int_0^{2\pi} \sin nx \cdot \cos nx \, dx = 0$

(ix) $[uv]_1 = uv_1 - u'v_2 = u''v_3 - u'''v_4 + \ldots$

where $v_1 = \int v \, dx$, $v_2 = \int v_1 \, dx$ and so on $u' = \frac{du}{dx}$, $u'' = \frac{d^2u}{dx^2}$ and so on and (x) $\sin n\pi = 0$, $\cos n\pi = (-1)^n$ where $n \in I$

## 10.7 DETERMINATION OF FOURIER COEFFICIENTS (EULER'S FORMULAE)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + \ldots + b_n \sin nx + \ldots \quad \ldots(1)$$

(i) **To find $a_0$:** Integrate both sides of (1) from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) dx = \frac{a_0}{2}\int_0^{2\pi} dx + a_1 \int_0^{2\pi} \cos x \, dx + a_2 \int_0^{2\pi} \cos 2x \, dx + \ldots + a_n \int_0^{2\pi} \cos nx \, dx + \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x \, dx + b_2 \int_0^{2\pi} \sin 2x \, dx + \ldots + b_n \int_0^{2\pi} \sin nx \, dx + \ldots$$
$$= \frac{a_0}{2}\int_0^{2\pi} dx \quad \text{(other integrals = 0 by formulae (i) and (ii) of Art 10.6)}$$
$$\int_0^{2\pi} f(x) dx = \frac{a_0}{2} 2\pi \Rightarrow \boxed{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) dx} \quad \ldots(2)$$

(ii) **To find $a_n$:** Multiply each side of (1) by $\cos nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) \cos nx \, dx = \frac{a_0}{2}\int_0^{2\pi} \cos nx \, dx + a_1 \int_0^{2\pi} \cos x \cos nx \, dx + \ldots + a_n \int_0^{2\pi} \cos^2 nx \, dx \ldots$$
$$+ b_1 \int_0^{2\pi} \sin x \cos nx \, dx + b_2 \int_0^{2\pi} \sin 2x \cos nx \, dx + \ldots$$



$$= a_n \int_0^{2\pi} \cos^2 nx \, dx = a_n \pi \quad \text{(Other integrals = 0, by formulae Art. 10.6)}$$

$$\therefore \quad \boxed{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx} \quad \ldots(3)$$

By taking $n = 1, 2 \ldots$ we can find the values of $a_1, a_2 \ldots$.

(iii) **To find $b_n$:** Multiply each side of (1) by $\sin nx$ and integrate from $x = 0$ to $x = 2\pi$.

$$\int_0^{2\pi} f(x) \sin nx \, dx = \frac{a_0}{2}\int_0^{2\pi} \sin nx \, dx + a_1 \int_0^{2\pi} \cos x \sin nx \, dx + \ldots + a_n \int_0^{2\pi} \cos nx \sin nx \, dx +$$
$$\ldots + b_1 \int_0^{2\pi} \sin x \sin nx \, dx + \ldots + b_n \int_0^{2\pi} \sin^2 nx \, dx + \ldots$$
$$= b_n \int_0^{2\pi} \sin^2 nx \, dx \quad \text{(All other integrals = 0, Article No. 10.5)}$$
$$= b_n \pi$$

$$\therefore \quad \boxed{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx} \quad \ldots(4)$$

**Note:** To get similar formula of $a_0$, $\frac{1}{2}$ has been written with $a_0$ in Fourier series.

**Example 1.** *Find the Fourier series representing*

$$f(x) = x, \quad 0 < x < 2\pi$$

*and sketch its graph from $x = -4\pi$ to $x = 4\pi$.*

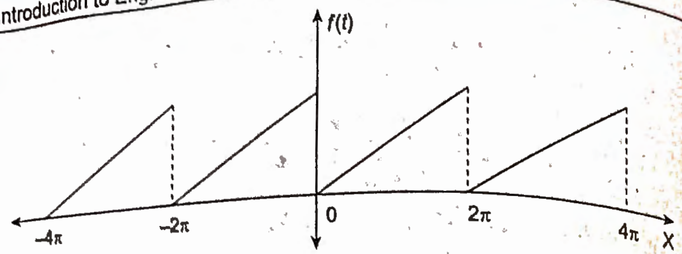
**Solution.** Let $f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots \quad \ldots(1)$

Hence $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x) dx = \frac{1}{\pi}\int_0^{2\pi} x \, dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \cos nx \, dx$$
$$= \frac{1}{\pi}\left[x \frac{\sin nx}{n} - 1 \cdot \left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{\cos 2n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{1}{n^2\pi}(1-1) = 0$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \, dx$$
$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1 \cdot \left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-2\pi \cos 2n\pi}{n}\right] = -\frac{2}{n}$$

Substituting the values of $a_0, a_1, a_2 \ldots, b_1, b_2 \ldots$ in (1), we get

$$f(x) = \pi - 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right] \quad \textbf{Ans.}$$




**Example 2.** *Given that $f(x) = x + x^2$ for $-\pi < x < \pi$, find the Fourier expression of $f(x)$.*

*Deduce that* $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ (Uttarakhand, June 2009)

**Solution.** Let $x + x^2 = \dfrac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots$ ...(1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (2x+1)\frac{(-\cos nx)}{n^2} + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(2\pi+1)\frac{\cos n\pi}{n^2} - (-2\pi+1)\frac{\cos(-n\pi)}{n^2}\right] = \frac{1}{\pi}\left[4\pi\frac{\cos n\pi}{n^2}\right] = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(-\frac{\cos nx}{n}\right) - (2x+1)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) + 2\left(\frac{\cos n\pi}{n^3}\right) + (-\pi+\pi^2)\left(\frac{\cos n\pi}{n}\right) - 2\left(\frac{\cos n\pi}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos n\pi\right] = -\frac{2}{n}(-1)^n$$

Substituting the values of $a_0$, $a_n$, $b_n$, in (1), we get

$$x + x^2 = \frac{\pi^2}{3} + 4\left[-\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \ldots\right] - 2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \ldots\right] \quad \ldots(2)$$

Putting $x = \pi$, (2) becomes $\pi + \pi^2 = \dfrac{\pi^2}{3} + 4\left[1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots\right]$ ...(3)



Putting $x = -\pi$, (2) becomes $-\pi + \pi^2 = \dfrac{\pi^2}{3} + 4\left[1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \ldots\right]$ ...(4)

Adding (3) and (4),

$$2\pi^2 = \frac{2\pi^2}{3} + 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\frac{4\pi^2}{3} = 8\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right]$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$ **Ans.**

**Example 3.** *Find the Fourier series expansion for $f(x) = x + \dfrac{x^2}{4}$, $-\pi \le x \le \pi$*

(U.P. II Semester, 2009)

**Solution.** Let $x + \dfrac{x^2}{4} = \dfrac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots\ldots + b_1 \sin x + b_2 \sin 2x + \ldots\ldots$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{-\pi}^{\pi}\left(x + \dfrac{x^2}{4}\right)dx$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{12}\right]_{-\pi}^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{12} - \frac{\pi^2}{2} + \frac{\pi^3}{12}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{12}\right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x + \frac{x^2}{4}\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x + \frac{x^2}{4}\right)\left(\frac{\sin nx}{n}\right) - \left(1 + \frac{2x}{4}\right)\left(\frac{-\cos nx}{n^2}\right) + \frac{1}{2}\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\pi + \frac{\pi^2}{4}\right)\left(\frac{\sin n\pi}{n}\right) + \left(1 + \frac{2\pi}{4}\right)\left(\frac{\cos n\pi}{n^2}\right) - \frac{1}{2}\left(\frac{\sin n\pi}{n^3}\right) - \left(-\pi + \frac{\pi^2}{4}\right)\left(\frac{\sin(-n\pi)}{n}\right)\right.$$
$$\left. - \left(1 - \frac{2\pi}{4}\right)\frac{\cos(-n\pi)}{n^2} + \frac{1}{2}\frac{\sin(-n\pi)}{n^3}\right]$$

$$= \frac{1}{\pi}\left[\left(1 + \frac{\pi}{2}\right)\frac{(-1)^n}{n^2} - \left(1 - \frac{\pi}{2}\right)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{n^2\pi}\left[1 + \frac{\pi}{2} - 1 + \frac{\pi}{2}\right] = \frac{(-1)^n}{n^2\pi}(\pi) = \frac{(-1)^n}{n^2}$$

$a_1 = -1$

$a_2 = \dfrac{1}{4}$

$a_3 = -\dfrac{1}{9}$

$$a_4 = \frac{1}{16}$$

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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+\frac{x^2}{4}\right)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{4}\sin nx\, dx$$

Even function    Odd function

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx + 0$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\frac{\cos n\pi}{n}+\frac{\sin n\pi}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{2(-1)^n}{n}$$

$$b_1 = \frac{2}{1},\quad b_2 = -1,\quad b_3 = \frac{2}{3},\quad b_4 = -\frac{1}{2}\ldots$$

Hence, Fourier series of the given function.

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{\pi^2}{12}+\sum\frac{(-1)^n}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{2}{n}(-1)^n\sin nx$$

$$f(x) = \frac{\pi^2}{12}-\cos x+\frac{1}{4}\cos 2x-\frac{1}{9}\cos 3x+\frac{1}{16}\cos 4x+\ldots+2\sin x-\sin 3x$$

$$+\frac{2}{3}\sin 4x-\frac{1}{2}\sin 4x+\ldots$$

**Ans.**

## EXERCISE 10.1

1. Find a Fourier series to represent, $f(x) = \pi - x$ for $0 < x < 2\pi$

**Ans.** $2\left[\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\ldots+\frac{1}{n}\sin nx+\ldots\right]$



2. Find a Fourier series to represent $x - x^2$ from $x = -\pi$ to $\pi$ and show that

$$\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

**Ans.** $-\frac{\pi^2}{3}+4\left[\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+\ldots\right]+2\left[\frac{\sin x}{1^2}-\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}-\frac{\sin 4x}{4^2}+\ldots\right]$

3. Find a Fourier series to represent: $f(x) = x \sin x$, for $0 < x < 2\pi$.

**Ans.** $-1+\pi\sin x-\frac{1}{2}\cos x+2\left[\frac{\cos 2x}{2^2-1}+\frac{\cos 3x}{3^2-1}+\frac{\cos 4x}{4^2-1}+\ldots\right]$

4. Find a Fourier series to represent the function $f(x) = e^x$, for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$.

**Ans.** $\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{1^2+1}\cos x+\frac{1}{2^2+1}\cos 2x-\frac{1}{3^2+1}\cos 3x+\ldots\right.$

$\left.+\frac{1}{1^2+1}\sin x-\frac{2}{2^2+1}\sin 2x-\frac{3}{3^2+1}\sin 3x\ldots\right]$

$$\frac{\pi}{\sinh\pi} = 1+2\left[-\frac{1}{2}+\frac{1}{5}-\frac{1}{10}+\ldots\right]$$

5. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 \le x < 2\pi$.

**Ans.** $\frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2}+\frac{1}{2}\cos x+\frac{1}{5}\cos 2x+\frac{1}{10}\cos 3x+\frac{1}{2}\sin x+\frac{2}{5}\sin 2x+\frac{3}{10}\sin 3x+\ldots\right]$

6. If $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$, show that $f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

7. Prove that $x^2 = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$, $-\pi < x < \pi$.

Hence show that

(i) $\sum\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$    (ii) $\sum\frac{1}{n^4} = \frac{\pi^4}{90}$

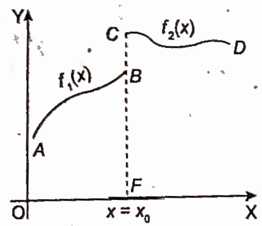
8. If $f(x)$ is a periodic function defined over a period $(0, 2\pi)$ by $f(x) = \frac{(3x^2-6x\pi+2\pi^2)}{12}$.

Prove that $f(x) = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ and hence show that $\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}\ldots$

## 10.8 FOURIER SERIES FOR DISCONTINUOUS FUNCTIONS

Let the function $f(x)$ be defined by

$$f(x) = \begin{cases} f_1(x), & c < x < x_0 \\ f_2(x), & x_0 < x < c+2\pi \end{cases}$$




where $x_0$ is the point of discontinuity in the interval $(c, c+2\pi)$. In such cases also, we obtain the Fourier series for $f(x)$ in the usual way. The values of $a_0$, $a_n$, $b_n$ are evaluated by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\,dx + \int_{x_0}^{c+2\pi} f_2(x)\,dx\right];$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\cos nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0} f_1(x)\sin nx\,dx + \int_{x_0}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

If $x = x_0$ is the point of finite discontinuity, then the sum of the Fourier series

$$= \frac{1}{2}\left[\lim_{h\to 0} f(x_0 - h) + \lim_{h\to 0} f(x_0 + h)\right]$$

$$= \frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right] = \frac{1}{2}(FB + FC)$$

**Remarks.**

1. It may be seen from the graph, that at a point of finite discontinuity $x = x_0$, there is a finite jump equal to $BC$ in the value of the function $f(x)$ at $x = x_0$.
2. A given function $f(x)$ may be defined by different formulae in different regions. Such types of functions are quite common in Fourier Series.
3. At a point of discontinuity the sum of the series is equal to the mean of the limits on the right and left.

## 10.9 FUNCTION DEFINED IN TWO OR MORE SUB-RANGES

**Example 4.** *Find the fourier series to represent the function f(x) given by:*

$$f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$$

*Hence show that:* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$ *(U.P. II Semester 2010)*

**Solution.** $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_0^{\pi} k\,dx\right] = \frac{1}{\pi}\left[[-kx]_{-\pi}^{0} + [kx]_0^{\pi}\right]$$

$$= \frac{1}{\pi}k[0 - \pi + \pi - 0] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_0^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}k\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0} + \left\{\frac{\sin nx}{n}\right\}_0^{\pi}\right] = \frac{1}{\pi}k[-0+0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$



$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_0^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}k\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}k\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}k\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

If $n$ is even $b_n = 0$

If $n$ is odd $b_n = \frac{4k}{n\pi}$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(1)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \ldots$$

Thus required Fourier sine series is

$$f(x) = \frac{4k}{\pi}\sin x + \frac{4k}{3\pi}\sin 3x + \frac{4k}{5\pi}\sin 5x + \ldots$$

$$\Rightarrow \quad f(x) = \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right] \quad \ldots(2)$$

Putting $x = \frac{\pi}{2}$ in (2), we get

$$k = \frac{4k}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\right]$$

$$\Rightarrow \quad 1 = \frac{4}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots\right]$$

$$= \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$$

$$\Rightarrow \quad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

**Proved.**

**Example 5.** *Find the Fourier series of the function*

$$f(x) = \begin{cases} -1 & \text{for } -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for } \frac{\pi}{2} < x < \pi. \end{cases}$$

**Solution.** Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots \quad \ldots(1)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}0\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}1\,dx$$

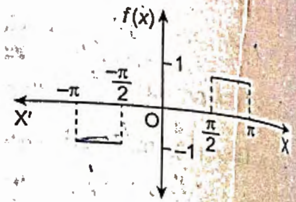
$$= \frac{1}{\pi}[-x]_{-\pi}^{-\pi/2} + \frac{1}{\pi}[x]_{\pi/2}^{\pi} = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\pi-\frac{\pi}{2}\right] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\cos nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{-\pi/2} + \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] + \frac{1}{\pi}\left[-\frac{\sin\frac{n\pi}{2}}{n}\right] = 0$$




$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(0)\sin nx\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi}(1)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{-\pi/2} - \frac{1}{\pi}\left[\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right] + \frac{1}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

$$b_1 = \frac{2}{\pi}, \quad b_2 = -\frac{2}{\pi}, \quad b_3 = \frac{2}{3\pi}$$

Putting the values of $a_0$, $a_n$, $b_n$ in (1), we get

$$f(x) = \frac{1}{\pi}\left[2\sin x - 2\sin 2x + \frac{2}{3}\sin 3x + \ldots\right]$$ **Ans.**

**Example 6.** *Find the Fourier series for the periodic function*

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

**Solution.** Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$ ... (1)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$



$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = -\frac{2}{n^2\pi}, \text{ when } n \text{ is odd}$$

$= 0$, when $n$ is even.

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n}\right] = -\frac{(-1)^n}{n}$$

Substituting the values of $a_0$, $a_1$, $a_2$ ... $b_1$, $b_2$, ... in (1), we get

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$ **Ans.**

## 10.10 DISCONTINUOUS FUNCTIONS

At a point of discontinuity, Fourier series gives the value of $f(x)$ as the arithmetic mean of left and right limits.

At the point of discontinuity, $x = c$

$$\text{At } x = c, \quad f(x) = \frac{1}{2}[f(c-0) + f(c+0)]$$

**Example 7.** *Find the Fourier series for f(x), if* $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

*Deduce that* $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$.

**Solution.** Let $f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + \ldots + a_n\cos nx + \ldots$

$+ b_1\sin x + b_2\sin 2x + \ldots + b_n\sin nx + \ldots$ ... (1)

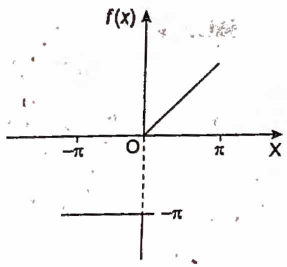
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Then $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + (x^2/2)_0^{\pi}\right] = \frac{1}{\pi}(-\pi^2 + \pi^2/2) = -\frac{\pi}{2}$;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right] = \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}(\cos n\pi - 1)$$




$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{\pi\cos nx}{n}\right)_{-\pi}^{0}+\left(-x\frac{\cos nx}{n}+\frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi)-\frac{\pi}{n}\cos n\pi\right]=\frac{1}{n}(1-2\cos n\pi)$$

$$f(x)=-\frac{\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right)+3\sin x-\frac{\sin 2x}{2}+\frac{3\sin 3x}{3}-\frac{\sin 4x}{4} \quad \dots(2)$$

Putting $x = 0$ in (2), we get $f(0)=-\frac{\pi}{4}-\frac{2}{\pi}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\infty\right)$ ...(3)

Now, $f(x)$ is discontinuous at $x = 0$.

But $f(0-0) = -\pi$ and $f(0+0) = 0$

$$\therefore \quad f(0)=\frac{1}{2}\left[f(0-0)+f(0+0)\right]=-\pi/2$$

Form (3), $-\frac{\pi}{2}=-\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right] \Rightarrow \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$ **Proved.**

**Example 8.** *Obtain Fourier Series of the function* $f(x)=\begin{cases} x; & -\pi<x<0 \\ -x, & 0<x<\pi \end{cases}$

*and hence show that* $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$ (U.P., II Semester, June 2008)

**Solution.** We have, $f(x)=\begin{cases} x; & -\pi<x<0 \\ -x, & 0<x<\pi \end{cases}$

Here $f(x)$ is an even function so $b_n = 0$

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}-x\,dx=-\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=-\frac{2}{\pi}\left[\frac{\pi^2}{2}\right]=-\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}-x\cos nx\,dx=-\frac{2}{\pi}\left[x\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$=-\frac{2}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]=\frac{2}{\pi n^2}\left[1-(-1)^n\right]=\begin{cases} 0, & n \text{ is even} \\ \frac{4}{\pi n^2}, & n \text{ is odd} \end{cases}$$

Fourier series

$$f(x)=-\frac{\pi}{2}+\frac{4}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right]$$ **Ans.**

At the point of discontinuity

$$f(0)=\frac{1}{2}\left[f(0-0)+f(0+0)\right]=\frac{1}{2}(0-0)=0$$

Putting $x = 0$ in the above, we get

$$0=-\frac{\pi}{2}+\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots\right)$$

$$\Rightarrow \quad \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\dots=\frac{\pi^2}{8}$$ **Ans.**



**Example 9.** *Find the Fourier series expansion of the periodic function of period $2\pi$, defined by*

$$f(x)=\begin{cases} x, & \text{if } -\frac{\pi}{2}<x<\frac{\pi}{2} \\ \pi-x, & \text{if } \frac{\pi}{2}<x<\frac{3\pi}{2} \end{cases}$$

**Solution.** Let $f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\dots+b_1\sin x+b_2\sin 2x+\dots$ ... (1)

$$a_0=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}x\,dx+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\,dx=\frac{1}{\pi}\left(\frac{x^2}{2}\right)_{-\pi/2}^{\pi/2}+\frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\right)_{\pi/2}^{3\pi/2}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{8}-\frac{\pi^2}{8}\right)+\frac{1}{\pi}\left(\frac{3\pi^2}{2}-\frac{9\pi^2}{8}-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right)=\pi\left(\frac{3}{2}-\frac{9}{8}-\frac{1}{2}+\frac{1}{8}\right)=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}x\cos nx\,dx+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[x\frac{\sin nx}{n}-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi/2}^{\pi/2}+\frac{1}{\pi}\left[(\pi-x)\frac{\sin nx}{n}-(-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}+\frac{\cos\frac{n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}-\frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$+\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\sin\frac{3n\pi}{2}}{n}+\frac{\cos\frac{3n\pi}{2}}{n^2}-\frac{\pi}{2}\frac{\sin\frac{n\pi}{2}}{n}-\frac{\cos\frac{n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2n}\left(\sin\frac{3n\pi}{2}+\sin\frac{n\pi}{2}\right)-\frac{1}{n^2}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\sin n\pi\cos\frac{n\pi}{2}+\frac{2}{n^2}\sin\frac{n\pi}{2}\sin n\pi\right]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}x\sin nx\,dx+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi/2}x\sin nx\,dx+\frac{1}{\pi}\int_{\pi/2}^{3\pi/2}(\pi-x)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi/2}+\frac{1}{\pi}\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{3\pi/2}$$

$$=\frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]+\frac{1}{\pi}\left[\frac{\pi}{2}\frac{\cos\frac{3n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{\sin\frac{n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n}+\frac{3\sin\frac{n\pi}{2}}{n^2}+\frac{\pi}{2}\frac{\cos 3\frac{n\pi}{2}}{n}-\frac{\sin\frac{3n\pi}{2}}{n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi}{2n}\left(\cos\frac{3n\pi}{2}-\cos\frac{n\pi}{2}\right)+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]$$

$$=\frac{1}{\pi}\left[-\frac{\pi}{n}\sin\frac{n\pi}{2}\sin n\pi+\frac{3}{n^2}\sin\frac{n\pi}{2}-\frac{1}{n^2}\sin\frac{3n\pi}{2}\right]=\frac{1}{n^2\pi}\left[3\sin\frac{n\pi}{2}-\sin\frac{3n\pi}{2}\right]$$

Substituting the values of $a_0, a_1, a_2 \ldots b_1, b_2, \ldots$ in (1), we get

$$f(x)=\frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\ldots\right]$$ Ans.

**Example 10.** *Find the Fourier series of the function defined as*

$$f(x)=\begin{cases} x+\pi, & \text{for } 0\le x\le\pi, \\ -x-\pi & \text{for } -\pi\le x<0\end{cases} \quad \text{where } f(x+2\pi)=f(x).$$

**Solution.** $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\,dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\,dx=\frac{1}{\pi}\left(-\frac{x^2}{2}-\pi x\right)_{-\pi}^{0}+\frac{1}{\pi}\left(\frac{x^2}{2}+\pi x\right)_{0}^{\pi}$$

$$=\frac{1}{\pi}\left(\frac{\pi^2}{2}-\pi^2\right)+\frac{1}{\pi}\left(\frac{\pi^2}{2}+\pi^2\right)=\pi\left(\frac{1}{2}-1\right)+\pi\left(\frac{1}{2}+1\right)=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(-x-\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\frac{\sin nx}{n}-(-1)\left\{-\frac{\cos nx}{n^2}\right\}\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{1}{n^2}+\frac{(-1)^n}{n^2}\right]+\frac{1}{\pi}\left[\frac{(-1)^n}{n^2}-\frac{1}{n^2}\right]=\frac{2}{n^2\pi}[(-1)^n-1]=\frac{-4}{n^2\pi},\ \text{if } n \text{ is odd.}$$

$= 0$, if $n$ is even.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x-\pi)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}(x+\pi)\sin nx\,dx$$



$$=\frac{1}{\pi}\left[(-x-\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}+\frac{1}{\pi}\left[(x+\pi)\left(-\frac{\cos nx}{n}\right)-(-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{0}^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\pi}{n}\right]+\frac{1}{\pi}\left[-\frac{2\pi}{n}(-1)^n+\frac{\pi}{n}\right]=\frac{1}{n}[(1)-2(-1)^n+(1)]=\frac{2}{n}[1-(-1)^n]$$

$=\frac{4}{n}$, if $n$ is odd.

$= 0$, if $n$ is even.

Fourier series is

$$f(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\ldots+b_1\sin x+b_2\sin 2x+\ldots$$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right)+4\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\ldots\right)$$ Ans.

## EXERCISE 10.2

1. Find the constant term if the function $f(x) = x + x^2$ is expanded in Fourier series defined in $(-1, 1)$. *(GBTU, II Sem., Jan. 2012)*

   **Ans.** Constant term $=\frac{a_0}{2}=\frac{1}{3}$

2. Find the Fourier series of the function

   $$f(x)=\begin{cases}-1 & \text{for } -\pi<x<0\\ 1 & \text{for } 0<x<\pi\end{cases}$$ *(U.P. II Semester, 2013)*

   where $f(x + 2\pi) = f(x)$.

   **Ans.** $\frac{4}{\pi}\left[\frac{1}{1}\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\frac{1}{7}\sin 7x+\ldots\right]$

3. Find the Fourier series for the function

   $$f(x)=\begin{cases}-\frac{\pi}{4} & \text{for } -\pi<x<0\\ \frac{\pi}{4} & \text{for } 0<x<\pi\end{cases}$$ and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x + 2\pi)$ for all $x$.

   Deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$

   **Ans.** $\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\frac{\sin 7x}{7}+\ldots$

4. Find the Fourier series of the function

   $$f(x)=\begin{cases}0 & \text{for } -\pi\le x\le 0\\ 1 & \text{for } 0<x<\frac{\pi}{2}\\ 0 & \text{for } \frac{\pi}{2}\le x\le\pi\end{cases}$$

   **Ans.** $\frac{1}{4}+\frac{1}{\pi}\left[\frac{\cos x}{1}-\frac{\cos 3x}{3}+\frac{\cos 5x}{5}+\ldots+\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+\ldots\right]$

5. Obtain a Fourier series to represent the following periodic function

$$f(x)=\begin{cases}0, & 0<x<\pi\\ 1, & \pi<x<2\pi\end{cases}$$

**Ans.** $\frac{1}{2}-\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right)$

6. Find the Fourier expansion of the function defined in a single period by the relations,

$$f(x)=\begin{cases}1 & \text{for } 0<x<\pi\\ 2 & \text{for } \pi<x<2\pi\end{cases}$$

and from it deduce that $\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$ **Ans.** $\frac{3}{2}-\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right)$

7. Find a Fourier series to represent the function

$$f(x)=\begin{cases}0 & \text{for } -\pi<x\le 0\\ \frac{1}{4}\pi x & \text{for } 0<x<\pi\end{cases}$$

and hence deduce that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$

**Ans.** $\frac{\pi^2}{16}+\sum_{n=1}^{\infty}\left(\frac{[(-1)^n-1]}{4n^2}\cos nx-\frac{(-1)^n\pi}{4n}\sin nx+...\right)$

8. Find the Fourier series for $f(x)$, if

$$f(x)=\begin{cases}-\pi & \text{for } -\pi<x\le 0\\ x & \text{for } 0<x<\pi\\ \frac{-\pi}{2} & \text{for } x=0\end{cases}$$

Deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...=\frac{\pi^2}{8}$

**Ans.** $-\frac{\pi}{4}-\frac{2}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...\right)+3\sin x-\frac{1}{2}\sin 2x+\frac{3}{3}\sin 3x-\frac{1}{4}\sin 4x+...$

9. Obtain a Fourier series to represent the function $f(x)=|x|$ for $-\pi<x<\pi$ and hence deduce $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...$ **Ans.** $\frac{\pi}{2}-\frac{4}{\pi}\left[\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+...\right]$

10. Expand as a Fourier series, the function $f(x)$ defined as

$$f(x)=\begin{cases}\pi+x & \text{for } -\pi<x<-\frac{\pi}{2}\\ \frac{\pi}{2} & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2}\\ \pi-x & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$$

**Ans.** $\frac{3\pi}{8}+\frac{2}{\pi}\left[\frac{1}{1^2}\cos x-\frac{2}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+...\right]$

11. Obtain a Fourier series to represent the function

$f(x)=|\sin x|$ for $-\pi<x<\pi$

$\left\{\textbf{Hint } f(x)=\begin{cases}-\sin x & \text{for } -\pi<x<0\\ \sin x & \text{for } 0<x<\pi\end{cases}\right.$



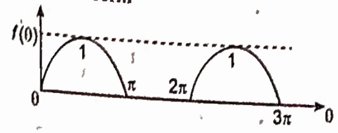
**Ans.** $\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos 2x+\frac{1}{15}\cos 4x+\frac{1}{35}\cos 6x+...\right]$

12. An alternating current after passing through a rectifier has the form

$I=I\sin\theta$ for $0<\theta<\pi$

$=0$ for $\pi<\theta<2\pi$

Find the Fourier series of the function.




**Ans.** $\frac{I}{\pi}-\frac{2I}{\pi}\left(\frac{\cos 2\theta}{3}+\frac{\cos 4\theta}{15}+...\right)+\frac{I}{2}\sin\theta$

13. If $f(x)=\begin{cases}0 & \text{for } -\pi<x<0\\ \sin x & \text{for } 0<x<\pi\end{cases}$

Prove that $f(x)=\frac{1}{\pi}+\frac{\sin x}{2}-\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}$. Hence show that $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}...\infty=\frac{1}{4}(\pi-2)$

## 10.11 EVEN FUNCTION AND ODD FUNCTION (T)

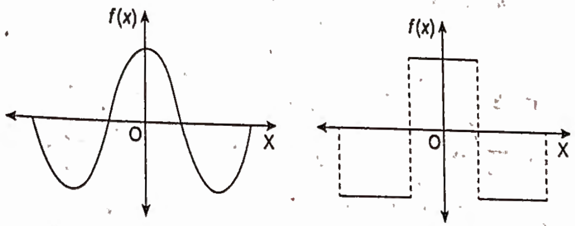
### (a) Even Function

A function $f(x)$ is said to be even (or symmetric) function if, $f(-x)=f(x)$

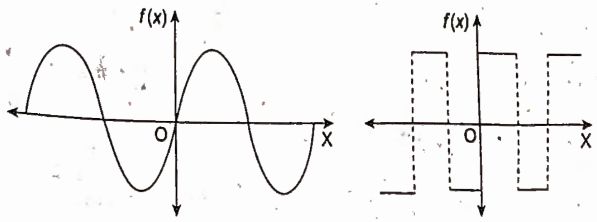
The graph of such a function is symmetrical with respect to $y$-axis [$f(x)$ axis]. Here $y$-axis is a mirror for the reflection of the curve.

The area under such a curve from $-\pi$ to $\pi$ is double the area from 0 to $\pi$.

$$\therefore \quad \int_{-\pi}^{\pi} f(x)\,dx=2\int_0^{\pi} f(x)\,dx$$




### (b) Odd Function




A function $f(x)$ is called odd (or skew symmetric) function if

$$f(-x)=-f(x)$$

Here the area under the curve from $-\pi$ to $\pi$ is zero.

$$\int_{-\pi}^{\pi} f(x)\,dx=0$$

**Expansion of an even function:**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

As $f(x)$ and $\cos nx$ are both even functions, therefore, the product of $f(x)$. $\cos nx$ is also an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

As $\sin nx$ is an odd function so $f(x)$. $\sin nx$ is also an odd function. We need not to calculate $b_n$. It saves our labour a lot.

The series of the even function will contain only cosine terms. (U.P. II Semester 2010)

**Expansion of an odd function:**

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

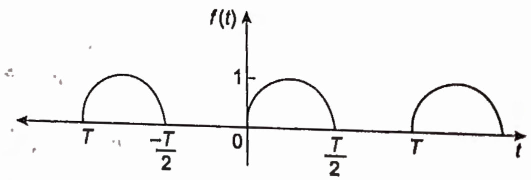
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad [f(x).\ \cos nx \text{ is odd function}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$[f(x).\ \sin nx$ is even function$]$

The series of the odd function will contain only sine terms.

The function shown below is neither odd nor even so it contains both sine and cosine terms.




**Example 11.** *Find the Fourier series expansion of the periodic function of period $2\pi$*

$f(x)=x^2, -\pi \le x \le \pi$. *Hence, find the sum of the series* $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}$

**Solution.** $f(x) = x^2,\ -\pi \le x \le \pi.$

This is an even function. $\therefore\ b_n = 0$

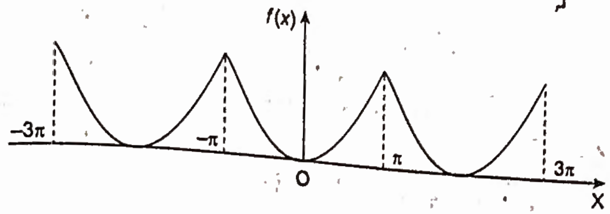
$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$



$$= \frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n}+\frac{2\pi\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}\right] = \frac{4(-1)^n}{n^2}$$




Fourier Series is $\quad f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \ldots + a_n\cos nx + \ldots$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

On putting $x = 0$, we have

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots\right]$$

$$\Rightarrow \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2}\ldots = \frac{\pi^2}{12}$$

**Ans.**

**Example 12.** *Obtain a Fourier expression for $f(x) = x^3$ for $-\pi < x < \pi$.*

**Solution.** $f(x) = x^3$ is an odd function.

$\therefore \quad a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3\sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\ \cos n\pi}{n^3}\right] = 2\cdot(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x + \ldots\right] \quad \textbf{Ans.}$$

**Example 13.** *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Hence deduce that* $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \ldots = \frac{\pi - 2}{4}$

(U.P., II Semester, Summer 2008, Uttarakhand, II Semester, June 2007)

**Solution.** $f(x) = x\sin x$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx \qquad \text{(Here } x\sin x \text{ is an even function)}$$

$$= \frac{2}{\pi}[x(-\cos x)-(1)(-\sin x)]_0^\pi = \frac{2}{\pi}(\pi) = 2$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin x\cos n x\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\{\sin(n+1)x - \sin(n-1)x\}\,dx$$

$$= \frac{1}{\pi}\int_0^\pi x\sin(n+1)\,x\,dx - \frac{1}{\pi}\int_0^\pi x\sin(n-1)\,x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1}\right)_0^\pi - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2}\right\}\right]_0^\pi$$

$$-\frac{1}{\pi}\left[x\left(-\frac{\cos(n-1)x}{(n-1)}\right) - (1)\left\{-\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\pi\frac{(-1)^{(n+1)}}{n+1}+0\right] - \frac{1}{\pi}\left[-\pi\frac{(-1)^{(n-1)}}{n-1}-0\right]$$

$$= -\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^{n+1}\left[-\frac{1}{n+1}+\frac{1}{n-1}\right] = \frac{2(-1)^{n+1}}{n^2-1}$$

$$a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi = \frac{1}{\pi}\left[-\frac{\pi}{2}\right] = -\frac{1}{2}$$

$b_n = 0$ [As $x \sin x \sin n x$ is an odd function]

Hence $f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nx = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{(n-1)(n+1)}\cos nx$

$$x\sin x = 1 + 2\left[-\frac{1}{4}\cos x - \frac{1}{1.3}\cos 2x + \frac{1}{2.4}\cos 3x - \frac{1}{3.5}\cos 4x + \ldots\right]$$

Putting $x = \frac{\pi}{2}$, we get $\frac{\pi}{2} = 1 + 2\left\{\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\right\}$

or $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \ldots$

⇒ $\frac{\pi}{4} - \frac{1}{2} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots$

⇒ $\frac{\pi-2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots$ **Proved.**



**Example 14.** *Find the Fourier Series expansion for the function*

$$f(x) = x\cos x,\ -\pi < x < \pi.$$

**Solution.** Since $x \cos x$ is an odd function therefore, $a_0 = a_n = 0$.

Let $x\cos x = \Sigma b_n \sin bx$, where

$$b_n = \frac{2}{\pi}\int_0^\pi x\cos x\cdot\sin nx\,dx = \frac{1}{\pi}\int_0^\pi x\{\sin(n+1)x+\sin(n-1)x\}dx$$

$$= \frac{1}{\pi}\int_0^\pi x\sin\,(n+1)\,x\,dx + \frac{1}{\pi}\int_0^\pi x\sin\,(n-1)\,x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right)+\frac{\sin(n+1)x}{(n+1)^2}\right]_0^\pi + \frac{1}{\pi}\left[-x\frac{\cos(n-1)x}{n-1}+\frac{\sin(n-1)x}{(n-1)^2}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[x\cdot\left\{-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{(n-1)}\right\} + 1\cdot\left\{\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\pi\cdot\left\{-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{(n-1)^2}\right\}\right]$$

$$\Rightarrow b_n = \left\{-\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right\},\ n \neq 1$$

$$b_n = -(-1)^{n+1}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$= -\left\{\frac{1}{(n+1)}+\frac{1}{(n-1)}\right\} = \frac{-2n}{n^2-1}, \quad \text{if } n \text{ is odd; } n \neq 1.$$

But $b_n = \left\{\frac{1}{n+1}+\frac{1}{n-1}\right\} = \frac{2n}{n^2-1}$, if $n$ is even; $n \neq 1$

If $n = 1$, then $b_1 = \frac{2}{\pi}\int_0^\pi x\cos x\cdot\sin x\,dx = \frac{1}{\pi}\int_0^\pi x\cdot\sin 2x\,dx$

$$= \frac{1}{\pi}\left[x\cdot\left(-\frac{\cos 2x}{2}\right) - 1\left(-\frac{\sin 2x}{4}\right)\right]_0^\pi = \frac{1}{\pi}\left[\pi\left(-\frac{1}{2}\right)\right] = -\frac{1}{2}$$

∴ $x\cos x = b_1\sin x + b_2\sin 2x + b_3\sin 3x + \ldots = -\frac{1}{2}\sin x + \frac{4\sin 2x}{2^2-1} - \frac{6\sin 3x}{3^2-1} + \ldots$ **Ans.**

## 10.12 HALF-RANGE SERIES, PERIOD 0 TO π

The given function is defined in the interval $(0, \pi)$ and it is immaterial whatever the function may be outside the interval $(0, \pi)$. To get the series of cosines only we assume that $f(x)$ is an even function in the interval $(-\pi, \pi)$.

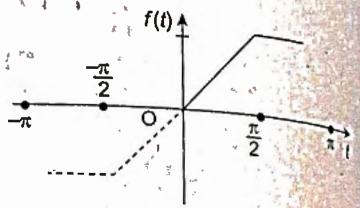
$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx \text{ and } b_n = 0$$

To expand $f(x)$ as a sine series we extend the function in the interval $(-\pi, \pi)$ as an odd function.

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx \text{ and } a_n = 0$$

**Example 15.** *Represent the following function by a Fourier sine series*

$$f(t) = \begin{cases} t, & 0 < t < \dfrac{\pi}{2} \\ \dfrac{\pi}{2}, & \dfrac{\pi}{2} < t < \pi \end{cases}$$




**Solution.** $b_n = \frac{2}{\pi}\int_0^\pi f(t)\sin nt\,dt$

$$= \frac{2}{\pi}\int_0^{\pi/2} t\sin nt\,dt + \frac{2}{\pi}\int_{\pi/2}^{\pi} \frac{\pi}{2}\sin nt\,dt$$

$$= \frac{2}{\pi}\left[t\left(-\frac{\cos nt}{n}\right) - (1)\left(-\frac{\sin nt}{n^2}\right)\right]_0^{\pi/2} + \frac{2}{\pi}\frac{\pi}{2}\left[-\frac{\cos nt}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{n\pi}{2}}{n} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \left[-\frac{\cos n\pi}{n} + \frac{\cos\frac{n\pi}{2}}{n}\right]$$

$$b_1 = \frac{2}{\pi}\left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] + \left[-\cos\pi + \cos\frac{\pi}{2}\right] = \frac{2}{\pi}[0+1] + [1] = \frac{2}{\pi} + 1$$

$$b_2 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\pi}{2} + \frac{\sin\pi}{2^2}\right] + \left[-\frac{\cos 2\pi}{2} + \frac{\cos\pi}{2}\right] = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{(-1)}{2} + 0\right] + \left[-\frac{1}{2} - \frac{1}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4}\right] - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[-\frac{\pi}{2}\frac{\cos\frac{3\pi}{2}}{3} + \frac{\sin\frac{3\pi}{2}}{3^2}\right] + \left[-\frac{\cos 3\pi}{3} + \frac{\cos\frac{3\pi}{2}}{3}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2}(0) - \frac{1}{9}\right] + \left[\frac{1}{3} + 0\right] = -\frac{2}{9\pi} + \frac{1}{3}$$

$$f(t) = \left(\frac{2}{\pi} + 1\right)\sin t - \frac{1}{2}\sin 2t + \left(-\frac{2}{9\pi} + \frac{1}{3}\right)\sin 3t + \ldots$$ **Ans.**

**Example 16.** *Find the Fourier sine series for the function $f(x) = e^{ax}$ for $0 < x < \pi$ where a is constant.*

**Solution.** $b_n = \frac{2}{\pi}\int_0^\pi e^{ax}\sin nx\,dx$ $\quad\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]\right]$



$$= \frac{2}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{e^{a\pi}}{a^2+n^2}(a\sin n\pi - n\cos n\pi) + \frac{n}{a^2+n^2}\right]$$

$$= \frac{2}{\pi}\frac{n}{a^2+n^2}\left[-(-1)^n e^{a\pi} + 1\right] = \frac{2n}{(a^2+n^2)\pi}\left[1 - (-1)^n e^{a\pi}\right]$$

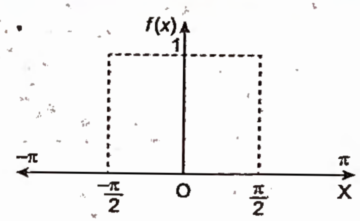
$$b_1 = \frac{2(1+e^{a\pi})}{(a^2+1^2)\pi}, \qquad b_2 = \frac{2.2.(1-e^{a\pi})}{(a^2+2^2)\pi}$$

$$e^{ax} = \frac{2}{\pi}\left[\frac{1+e^{a\pi}}{a^2+1^2}\sin x + \frac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x + \ldots\right]$$ **Ans.**

## EXERCISE 10.3

1. Find the Fourier cosine series for the function

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for } \dfrac{\pi}{2} < x < \pi. \end{cases}$$




**Ans.** $\frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots\right]$

2. Find a series of cosine of multiples of $x$ which will represent $f(x)$ in $(0, \pi)$ where

$$f(x) = \begin{cases} 0 & \text{for } 0 < x < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases}$$

Deduct that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots.\infty = \frac{\pi}{4}$ **Ans.** $\frac{\pi}{4} - \cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \ldots$

3. Express $f(x) = x$ as a sine in $0 < x < \pi$. **Ans.** $2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right]$

4. Find the cosine series for $f(x) = \pi - x$ in the interval $0 < x < \pi$.

**Ans.** $\frac{\pi}{2} + \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

5. If $f(x) = \begin{cases} x, & \text{for } 0 < x < \dfrac{\pi}{2} \\ \pi - x, & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases}$

Show that:

(i) $f(x)=\frac{4}{\pi}\left(\sin x-\frac{1}{3^2}\sin 3x+\frac{1}{5^2}\sin 5x-\ldots\right)$

(ii) $f(x)=\frac{\pi}{4}-\frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x+\frac{1}{3^2}\cos 6x+\frac{1}{5^2}\cos 10x+\ldots\right)$

6. Obtain the half-range cosine series for $f(x)=x^2$ in $0\le x\le\pi$.

**Ans.** $\frac{\pi^2}{3}-4\left(\cos x-\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x-\ldots\right)$

7. Find (i) sine series and (ii) cosine series for the function $f(x)=e^x$ for $0<x<\pi$.

**Ans.** (i) $\frac{2}{\pi}\sum_{1}^{\infty}n\left[\frac{1-(-1)^n e^{\pi}}{n^2+1}\right]\sin nx$ (ii) $\frac{e^{\pi}-1}{\pi}-\frac{2}{\pi}\sum_{1}^{\infty}\frac{1-(-1)^n e^{\pi}}{n^2+1}\cos nx$

8. If $f(x)=x+1$, for $0<x<\pi$, find its Fourier (i) sine series (ii) cosine series. Hence deduce that

(i) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$  (ii) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$

**Ans.** (i) $\frac{2}{\pi}\left[(\pi+2)\sin x-\frac{\pi}{2}\sin 2x+\frac{1}{3}(\pi+2)\sin 3x-\frac{\pi}{4}\sin 4x+\ldots\right]$

(ii) $\frac{\pi}{2}+1-4\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\ldots\right)$

9. Find the Fourier series expansion of the function

$f(x)=\cos(sx),-\pi\le x\le\pi$

Where $s$ is a fraction. Hence, show that $\cot\theta=\frac{1}{\theta}+\frac{2\theta}{\theta^2-\pi^2}+\frac{2\theta}{\theta^2-4\pi^2}+\ldots$

**Ans.** $\frac{\sin\pi x}{\pi s}+\frac{1}{\pi}\Sigma\left(\frac{\sin(s\pi+n\pi)}{s+n}+\frac{\sin(s\pi-n\pi)}{s-n}\right)\cos nx$

## 10.13 CHANGE OF INTERVAL AND FUNCTION HAVING ARBITRARY PERIOD

In electrical engineering problems, the period of the function is not always $2\pi$ but $T$ or $2c$. This period must be converted to the length $2\pi$. The independent variable $x$ is also to be changed proportionally.

Let the $f(x)$ be defined in the interval $(-c, c)$. Now we want to change the function to the period of $2\pi$ so that we can use the formulae of $a_n$, $b_n$ as discussed in Article 10.6.

$\because$ $2c$ is the interval for the variable $x$.

$\because$ 1 is the interval for the variable $=\frac{x}{2c}$

$\because$ $2\pi$ is the interval for the variable $=\frac{x\,2\pi}{2c}=\frac{\pi x}{c}$



So put $z=\frac{\pi x}{c}$ or $x=\frac{zc}{\pi}$

Thus the function $f(x)$ of period $2c$ is transformed to the function

$f\left(\frac{cz}{\pi}\right)$ or $F(z)$ of period $2\pi$.

$F(z)$ can be expanded in the Fourier series.

$$F(z)=f\left(\frac{cz}{\pi}\right)=\frac{a_0}{2}+a_1\cos z+a_2\cos 2z+\ldots+b_1\sin z+b_2\sin 2z+\ldots$$

where $a_0=\frac{1}{\pi}\int_0^{2\pi}F(z)\,dz=\frac{1}{\pi}\int_0^{2\pi}f\left(\frac{cz}{\pi}\right)dz$

$$=\frac{1}{\pi}\int_0^{2c}f(x)\,d\left(\frac{\pi x}{c}\right)=\frac{1}{c}\int_0^{2c}f(x)\,dx \qquad \left(\text{put } z=\frac{\pi x}{c}\right)$$

$$\boxed{a_0=\frac{1}{c}\int_0^{2c}f(x)\,dx}$$

$a_n=\frac{1}{\pi}\int_0^{2\pi}F(z)\cos nz\,dz=\frac{1}{\pi}\int_0^{2\pi}f\left(\frac{cz}{\pi}\right)\cos nz\,dz$

$$=\frac{1}{\pi}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,d\left(\frac{\pi x}{c}\right)=\frac{1}{c}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,dx. \qquad \left[\text{put } z=\frac{\pi x}{c}\right]$$

$$\boxed{a_n=\frac{1}{c}\int_0^{2c}f(x)\cos\frac{n\pi x}{c}\,dx}$$

Similarly, $$\boxed{b_n=\frac{1}{c}\int_0^{2c}f(x)\sin\frac{n\pi x}{c}\,dx}$$

**Example 17.** *Expand $f(x)=e^x$ in a cosine series over $(0, 1)$.*

**Solution.** Here, we have $f(x)=e^x$ and $c=1$

$$a_0=\frac{2}{c}\int_0^c f(x)\,dx=\frac{2}{1}\int_0^1 e^x\,dx=2(e-1)$$

$$a_n=\frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx=\frac{2}{1}\int_0^1 e^x\cos\frac{n\pi x}{1}\,dx$$

$$=2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi x+\cos n\pi x)\right]_0^1$$

$$=2\left[\frac{e^x}{n^2\pi^2+1}(n\pi\sin n\pi+\cos n\pi)-\frac{1}{n^2\pi^2+1}\right]$$

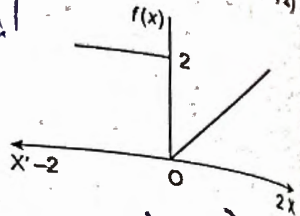
$$=\frac{2}{n^2\pi^2+1}\left[(-1)^n e-1\right]$$

$$f(x)=\frac{a_0}{2}+a_1\cos\pi x+a_2\cos 2\pi x+a_3\cos 3\pi x+\dots$$

$$e^x=e-1+2\left[\frac{-e-1}{\pi^2+1}\cos\pi x+\frac{e-1}{4\pi^2+1}\cos 2\pi x+\frac{-e-1}{9\pi^2+1}\cos 3\pi x+\dots\right]$$ **Ans.**

**Example 18.** *Find the Fourier series corresponding to the function f(x) defined in (−2, 2) as follows*

$$f(x)=\begin{cases}2 & \text{in } -2\le x\le 0\\ x & \text{in } 0<x<2\end{cases}$$




**Solution.** Here the interval is (−2, 2) and $c = 2$

$$a_0=\frac{1}{c}\int_{-c}^{c}f(x)\,dx=\frac{1}{2}\left[\int_{-2}^{0}2\,dx+\int_{0}^{2}x\,dx\right]$$

$$=\frac{1}{2}\left[[2x]_{-2}^{0}+\left(\frac{x^2}{2}\right)_0^2\right]=\frac{1}{2}[4+2]=3$$

$$a_n=\frac{1}{c}\int_{-c}^{c}f(x)\cos\left(\frac{n\pi x}{c}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}2\cos\frac{n\pi x}{2}dx+\int_{0}^{2}x\cos\frac{n\pi x}{2}dx\right]$$

$$=\frac{1}{2}\left[\frac{4}{n\pi}\left(\sin\frac{n\pi x}{2}\right)_{-2}^{0}+\left(x\frac{2}{n\pi}\sin\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)_0^2\right]$$

$$=\frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi-\frac{4}{n^2\pi^2}\right]=\frac{2}{n^2\pi^2}[(-1)^n-1]$$

$=-\frac{4}{n^2\pi^2}$, when $n$ is odd

$=0$, when $n$ is even.

$$b_n=\frac{1}{c}\int_{-c}^{c}f(x)\sin\frac{n\pi x}{c}dx=\frac{1}{2}\int_{-2}^{0}2\sin\frac{n\pi x}{2}dx+\frac{1}{2}\int_{0}^{2}x\sin\frac{n\pi x}{2}dx$$

$$=\frac{1}{2}\left[2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right]_{-2}^{0}+\frac{1}{2}\left[x\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)+(1)\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$=\frac{1}{2}\left[-\frac{4}{n\pi}+\frac{4}{n\pi}\cos n\pi\right]+\frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi+\frac{4}{n^2\pi^2}\sin n\pi\right]=\frac{1}{2}\left[-\frac{4}{n\pi}\right]=-\frac{2}{n\pi}$$

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+a_3\cos\frac{3\pi x}{c}+\dots+b_1\sin\frac{\pi x}{c}+b_2\sin\frac{2\pi x}{c}+b_3\sin\frac{3\pi x}{c}+\dots$$

$$=\frac{3}{2}-\frac{4}{\pi^2}\left\{\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\dots\right\}$$

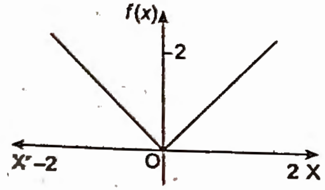
$$-\frac{2}{\pi}\left\{\frac{1}{1}\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\dots\right\}$$ **Ans.**



**Example 19.** *A period function of period 4 is defined as $f(x)=|x|, -2<x<2$. Find its Fourier series expansion.*

**Solution.** $f(x)=|x| \qquad -2<x<2$

$$\Rightarrow \quad f(x)=\begin{cases}x, & 0<x<2\\ -x, & -2<x<0\end{cases}$$




$$a_0=\frac{1}{c}\int_{-c}^{c}f(x)\,dx=\frac{1}{2}\int_0^2 x\,dx+\frac{1}{2}\int_{-2}^{0}(-x)\,dx$$

$$=\frac{1}{2}\left[\frac{x^2}{2}\right]_0^2+\frac{1}{2}\left[\frac{-x^2}{2}\right]_{-2}^0$$

$$=\frac{1}{4}(4-0)+\frac{1}{4}(0+4)=2$$

$$a_n=\frac{1}{c}\int_{-c}^{c}f(x)\cos\frac{n\pi x}{c}dx=\frac{1}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx+\frac{1}{2}\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}dx$$

$$=\frac{1}{2}\left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(-1)\left(-\frac{4}{n^2\pi^2}\right)\cos\frac{n\pi x}{2}\right]_{-2}^0$$

$$=\frac{1}{2}\left[0+\frac{4}{n^2\pi^2}(-1)^n-\frac{4}{n^2\pi^2}\right]+\frac{1}{2}\left[0-\frac{4}{n^2\pi^2}+\frac{4}{n^2\pi^2}(-1)^n\right]$$

$$=\frac{1}{2}\frac{4}{n^2\pi^2}[(-1)^n-1-1+(-1)^n]=\frac{4}{n^2\pi^2}[(-1)^n-1]$$

$=-\frac{8}{n^2\pi^2}$ (If $n$ is odd)

$=0$ (If $n$ is even)

$b_n=0$ as $f(x)$ is even function.

Fourier series is

$$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+\dots+b_1\sin\frac{\pi x}{c}+b_2\sin\frac{2\pi x}{c}+\dots$$

$$f(x)=1-\frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2}+\frac{\cos\frac{3\pi x}{2}}{3^2}+\frac{\cos\frac{5\pi x}{2}}{5^2}+\dots\right]$$ **Ans.**

**Example 20.** *Find the Fourier series for the function* $f(x)=\begin{cases}x; & 0<x<1\\ 1-x, & 1<x<2\end{cases}$

*(GBTU 2011)*

**Solution.** Here, we have $f(x)=\begin{cases} x, & 0<x<1 \\ 1-x, & 1<x<2 \end{cases}$

$$a_0=\frac{1}{c}\int_0^{2c} f(x)\,dx=\frac{1}{1}\left[\int_0^1 x\,dx+\int_1^2 (1-x)\,dx\right]$$

$$=\left[\left[\frac{x^2}{2}\right]_0^1+\left[x-\frac{x^2}{2}\right]_1^2\right]=\left[\frac{1}{2}-0+2-2-1+\frac{1}{2}\right]=0$$

$$a_n=\frac{1}{c}\int_0^{2c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx=\frac{1}{1}\left[\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx+\int_1^2 (1-x)\cos\left(\frac{n\pi x}{1}\right)dx\right]$$

$$=\left[x\frac{\sin n\pi x}{n\pi}-(1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1+\left[(1-x)\frac{\sin n\pi x}{n\pi}-(-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$=\left[\frac{\sin n\pi}{n\pi}+\left(\frac{\cos n\pi}{n^2\pi^2}\right)-\frac{1}{n^2\pi^2}\right]+\left[-\frac{\sin 2n\pi}{n\pi}-\frac{\cos 2n\pi}{n^2\pi^2}-0+\frac{\cos n\pi}{n^2\pi^2}\right]$$

$$=0+\frac{(-1)^n}{n^2\pi^2}-\frac{1}{n^2\pi^2}+0-\frac{1}{n^2\pi^2}+0\frac{(-1)^n}{n^2\pi^2}=\frac{2}{n^2\pi^2}[(-1)^n-1]$$

$$=-\frac{4}{n^2\pi^2},\ \text{when } n \text{ is odd}$$

$$=0,\ \text{when } n \text{ is even}$$

$$b_n=\frac{1}{c}\int_0^{2c} f(x)\sin\frac{n\pi x}{c}dx=\frac{1}{1}\int_0^1 x\sin\frac{n\pi x}{1}dx+\frac{1}{1}\int_1^2 (1-x)\sin\frac{n\pi x}{1}dx$$

$$=\frac{1}{1}\left[x\frac{(-\cos n\pi x)}{n\pi}-(1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1+\frac{1}{1}\left[(1-x)\left(\frac{-\cos n\pi x}{n\pi}\right)-(-1)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$=-\frac{\cos n\pi}{n\pi}+\frac{\sin n\pi}{n^2\pi^2}+\frac{\cos 2n\pi}{n\pi}-\frac{\sin 2n\pi}{n^2\pi^2}+0+\frac{\sin n\pi}{n^2\pi^2}$$

$$=-\frac{(-1)^n}{n\pi}+0+\frac{1}{n\pi}-0+0+0=\frac{1}{n\pi}[-(-1)^n+1]$$

$$=\frac{2}{n\pi},\ \text{if } n \text{ is odd}$$

$$=0,\ \text{if } n \text{ is even}$$

$$f(x)=\frac{a_0}{2}+a_1\cos\pi x+a_2\cos 2\pi x\ldots+b_1\sin\pi x+b_2\sin 2\pi x+\ldots$$

$$f(x)=-\frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\ldots\right)+\frac{2}{\pi}\left[\frac{\sin\pi x}{1}+\frac{\sin 3\pi x}{3}+\ldots\right]$$ **Ans.**

## 10.14 HALF PERIOD SERIES

**Cosine series:** $$f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+\ldots+a_n\cos\frac{n\pi x}{c}+\ldots$$



where $$a_0=\frac{2}{c}\int_0^c f(x)\,dx,\quad a_n=\frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

**Sine series:** $$f(x)=b_1\sin\frac{\pi x}{c}+b_2\sin\frac{2\pi x}{c}+\ldots+b_n\sin\frac{n\pi x}{c}+\ldots$$

where $$b_n=\frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx.$$

**Example 21.** *Expand for $f(x)=k$ for $0<x<2$ in a half range sine series.* (U.P., II Semester, June 2007)

**Solution.** $f(x)=k$

$$b_n=\frac{2}{c}\int_0^c f(x)\cdot\sin\frac{n\pi x}{c}dx \text{ in half range } (0,c)=\frac{2}{2}\int_0^2 k\sin\frac{n\pi x}{2}dx$$

$$=k\frac{2}{n\pi}\left(-\cos\frac{n\pi x}{2}\right)_0^2=\frac{2k}{n\pi}[-\cos n\pi+1]$$

Half range sine series is

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{2}\Rightarrow k=\sum_{n=1}^{\infty}\frac{2k}{n\pi}[1-\cos n\pi]\sin\frac{n\pi x}{2}$$ **Ans.**

**Example 22.** *Obtain the half-range sine series for the function $f(x)=x^2$ in the interval $0<x<3$.*

**Solution.** We know that half range sine series is given by $f(x)=\Sigma b_n \sin nx$

Where $b_n=\frac{2}{c}\int_0^c f(x)\sin\frac{n\pi x}{c}dx$ in the half-range $(0,c)$.

Here, we have half range $0<x<3$ and $f(x)=x^2$

$$b_n=\frac{2}{3}\int_0^3 x^2\sin\frac{n\pi x}{3}dx$$

$$=\frac{2}{3}\left[x^2\left(\frac{3}{n\pi}\right)\left(-\cos\frac{n\pi x}{3}\right)+2x\times\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\sin\frac{n\pi x}{3}-2\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\left(\frac{3}{n\pi}\right)\left(-\cos\frac{n\pi x}{3}\right)\right]_0^3$$

$$b_n=\frac{2}{3}\left[\left\{-\frac{27}{n\pi}(-1)^n-\frac{54}{n^3\pi^3}(-1)^n\right\}+\frac{54}{n^3\pi^3}\right]$$

$$b_n=\frac{2}{3}\left[\frac{54}{n^3\pi^3}\{1-(-1)^n\}-\frac{27}{n\pi}(-1)^n\right]\Rightarrow b_n=\frac{2}{3}\left[\frac{108}{n^3\pi^3}+\frac{27}{n\pi}\right] \text{ when } n \text{ is odd}$$

And $b_n=-\frac{18}{n\pi}$ when $n$ is even

∴ Half range sine series

$$f(x)=b_1\sin x+b_2\sin 2x+b_3\sin 3x+\ldots$$

$$=\frac{2}{3}\left[\frac{108}{\pi^3}\left(\frac{\sin x}{1^3}+\frac{\sin 3x}{3^3}+\frac{\sin 5x}{5^3}+\ldots\right)+\frac{27}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)\right]$$

$$-\frac{18}{\pi}\left(\frac{\sin 2x}{2}+\frac{\sin 4x}{4}+\ldots\right)$$ **Ans.**

**Example 23.** *Find the half range cosine series expansion of $f(x) = x - x^2$ in $0 < x < 1$.*
(GBTU, II Sem., Jan. 2012)

**Solution.** $f(x) = x - x^2$

$$a_0 = \frac{2}{C}\int_0^c f(x)\,dx = \frac{2}{1}\int_0^1 (x - x^2)\,dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

$$a_n = \frac{2}{C}\int_0^c f(x)\cos\frac{n\pi x}{c}\,dx = \frac{2}{1}\int_0^1 (x - x^2)\cos\frac{n\pi x}{1}\,dx$$

$$= 2\left[(x - x^2)\frac{\sin n\pi x}{nx} - (1 - 2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[0 - (-1)\left(\frac{-\cos n\pi}{n^2\pi^2}\right) + 0 + \left(\frac{-1}{n^2\pi^2}\right)\right]$$

$$= 2\left[\frac{-\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}\left[-(-1)^{n\pi} - 1\right] = \frac{-4}{n^2\pi^2}, \text{ when } n \text{ is even}$$

$= 0,$ when $n$ is odd.

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{c} + a_2\cos\frac{2\pi x}{c} + a_3\cos\frac{3\pi x}{c} + a_4\cos\frac{4n\pi}{c} + \ldots$$

$$x - x^2 = \frac{1}{6} + 0 - \frac{1}{\pi^2}\cos 2\pi x + 0 - \frac{1}{4\pi^2}\cos 4\pi x + \ldots$$

$$x - x^2 = \frac{1}{6} - \frac{1}{\pi^2}\left[\cos 2\pi x + \pi + \frac{1}{4}\cos 4\pi x + \ldots\right]$$ **Ans.**

**Example 24.** *Find the Fourier half-range cosine series of the function*

$$f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2 - t), & 1 < t < 2 \end{cases}$$

(U.P., II Semester, Summer 2007, 2006)

**Solution.** $f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 2(2 - t), & 1 < t < 2 \end{cases}$

Let $$f(t) = \frac{a_0}{2} + a_1\cos\frac{\pi t}{c} + a_2\cos\frac{2\pi t}{c} + a_3\cos\frac{3\pi t}{c} + \ldots$$

$$+ b_1\sin\frac{\pi t}{c} + b_2\sin\frac{2\pi t}{c} + b_3\sin\frac{3\pi t}{c} + \ldots \quad \ldots(1)$$

Here, $c = 2$, because it is half range series.

Hence, $$a_0 = \frac{2}{c}\int_0^c f(t)\,dt = \frac{2}{2}\int_0^1 2t\,dt + \frac{2}{2}2(2 - t)\,dt$$

$$= [t^2]_0^1 + \left[2\left(2t - \frac{t^2}{2}\right)\right]_1^2 = 1 + [(4t - t^2)]_1^2 = 1 + (8 - 4 - 4 + 1) = 2$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\frac{n\pi t}{c}\,dt = \frac{2}{2}\int_0^1 2t\cos\frac{n\pi t}{2}\,dt + \frac{2}{2}\int_1^2 2(2 - t)\cos\frac{n\pi t}{2}\,dt$$



$$= \left[2t\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_0^1$$

$$+ \left[(4 - 2t)\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right) - (2)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi t}{2}\right)\right]_1^2$$

$$= \left[\frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2}\right] + \left[0 - \frac{8}{n^2\pi^2}\cos n\pi - \frac{4}{n\pi}\sin\frac{n\pi}{2} + \frac{8}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$= \frac{16}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2}\cos\pi = \frac{8}{n^2\pi^2}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$f(t) = 1 + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]\cos\frac{n\pi t}{2}$$ **Ans.**

**Example 25.** *Obtain the Fourier cosine series expansion of the periodic function defined by $f(t) = \sin\left(\frac{\pi t}{l}\right)$, $0 < t < l$*

**Solution.** We have, $f(t) = \sin\left(\frac{\pi t}{l}\right), 0 < t < l$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)dt = \frac{2}{l}\left(-\frac{l}{\pi}\cos\frac{\pi t}{l}\right)_0^l = -\frac{2}{\pi}(\cos\pi - \cos 0) = -\frac{2}{\pi}(-1 - 1) = \frac{4}{\pi}$$

$$a_0 = \frac{2}{l}\int_0^l \sin\left(\frac{\pi t}{l}\right)\cos\frac{n\pi t}{l}\,dt = \frac{1}{l}\int_0^l\left[\sin\left(\frac{\pi t}{l} + \frac{n\pi t}{l}\right) - \sin\left(\frac{n\pi t}{l} - \frac{\pi t}{l}\right)\right]dt$$

$$= \frac{1}{l}\int_0^l \sin(n + 1)\frac{\pi t}{l}\,dt - \frac{1}{l}\int_0^l \sin(n - 1)\frac{\pi t}{l}\,dt$$

$$= \frac{1}{l}\left[-\frac{l}{(n + 1)\pi}\cos\frac{(n + 1)\pi t}{l}\right]_0^l - \frac{1}{l}\left[\frac{l}{(n - 1)\pi}\cos\frac{(n - 1)\pi t}{l}\right]_0^l$$

$$= \frac{-1}{(n + 1)\pi}[\cos(n + 1)\pi - \cos 0] + \frac{1}{(n - 1)\pi}[\cos(n - 1)\pi - \cos 0]$$

$$= \frac{1}{(n + 1)\pi}[(-1)^{n+1} - 1] + \frac{1}{(n - 1)\pi}[(-1)^{n+1} - 1]$$

$$= (-1)^{n+1}\left[\frac{1}{(n + 1)\pi} + \frac{1}{(n - 1)\pi}\right] + \frac{1}{(n + 1)\pi} - \frac{1}{(n - 1)\pi}$$

$$= (-1)^{n+1}\frac{2}{(n^2 - 1)\pi} - \frac{2}{(n^2 - 1)\pi} = \frac{2}{(n^2 - 1)\pi}[(-1)^{n+1} - 1]$$

$= \frac{-4}{(n^2 - 1)\pi}$ when $n$ is even

$= 0,$ when $n$ is odd.

The above formula for finding the value of $a_1$ is not applicable.

$$a_1=\frac{2}{l}\int_0^l \sin\frac{\pi t}{l}\cos\frac{\pi t}{l}\,dt=\frac{1}{l}\int_0^l \sin\frac{2\pi t}{l}\,dt$$

$$=\frac{1}{l}\left(-\frac{l}{2\pi}\cos\frac{2\pi t}{l}\right)_0^l=-\frac{l}{2\pi l}(\cos 2\pi-\cos 0)=-\frac{1}{2\pi}(1-1)=0$$

$$f(t)=\frac{a_0}{2}+a_1\cos\frac{\pi t}{l}+a_2\cos\frac{2\pi t}{l}+a_3\cos\frac{3\pi t}{l}+a_4\cos\frac{4\pi t}{l}+\ldots$$

$$=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{1}{3}\cos\frac{2\pi t}{l}+\frac{1}{15}\cos\frac{4\pi t}{l}+\frac{1}{35}\frac{6\pi t}{l}+\ldots\right]$$

**Ans.**

**Example 26.** *Find the Fourier cosine series expansion of the periodic function of period 1*

$$f(x)=\begin{cases}\frac{1}{2}+x, & -\frac{1}{2}<x\le 0\\ \frac{1}{2}-x, & 0<x<\frac{1}{2}\end{cases}$$

**Solution.** Let $f(x)=\frac{a_0}{2}+a_1\cos\frac{\pi x}{c}+a_2\cos\frac{2\pi x}{c}+\ldots$ ...(1)

as $f(x)$ is a cosine series.

Here $2c=1\Rightarrow c=\frac{1}{2}$

$$a_0=\frac{1}{c}\int_{-c}^{c} f(x)\,dx=\frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)dx+\frac{1}{1/2}\int_0^{1/2}\left(\frac{1}{2}-x\right)dx$$

$$=2\left[\frac{x}{2}+\frac{x^2}{2}\right]_{-1/2}^{0}+2\left[\frac{x}{2}-\frac{x^2}{2}\right]_0^{1/2}=2\left[\frac{1}{4}-\frac{1}{8}\right]+2\left[\frac{1}{4}-\frac{1}{8}\right]=\frac{1}{2}$$

$$a_n=\frac{1}{c}\int_{-c}^{c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$=\frac{1}{1/2}\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos\frac{n\pi x}{1/2}\,dx+\frac{1}{1/2}\int_0^{1/2}\left(\frac{1}{2}-x\right)\cos\frac{n\pi x}{1/2}\,dx$$

$$=2\int_{-1/2}^{0}\left(\frac{1}{2}+x\right)\cos 2n\pi x\,dx+2\int_0^{1/2}\left(\frac{1}{2}-x\right)\cos 2n\pi x\,dx$$

$$=2\left[\left(\frac{1}{2}+x\right)\frac{\sin 2n\pi x}{2n\pi}-(1)\left(-\frac{\cos 2n\pi x}{4n^2\pi^2}\right)\right]_{-1/2}^{0}$$

$$+2\left[\left(\frac{1}{2}-x\right)\frac{\sin 2n\pi x}{2n\pi}-(-1)\left(\frac{-\cos 2n\pi x}{4n^2\pi^2}\right)\right]_0^{1/2}$$

$$=2\left[0+\frac{1}{4n^2\pi^2}-\frac{(-1)^n}{4n^2\pi^2}\right]+2\left[0-\frac{(-1)^n}{4n^2\pi^2}+\frac{1}{4n^2\pi^2}\right]=\frac{1}{\pi^2}\left[\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right]$$



$$=\frac{2}{n^2\pi^2}\qquad\text{(if } n \text{ is odd)}$$

$$=0\qquad\text{(if } n \text{ is even)}$$

Substituting the values of $a_0, a_1, a_2, a_3, \ldots$ in (1), we have

$$f(x)=\frac{1}{4}+\frac{2}{\pi^2}\left[\frac{\cos 2\pi x}{1^2}+\frac{\cos 6\pi x}{3^2}+\frac{\cos 10\pi x}{5^2}+\ldots\right]$$

**Ans.**

**Example 27.** *Find the half range Fourier sine series of f(x) defined over the range $0<x<4$ as*

$$f(x)=\begin{cases}x, & 0<x<2\\ 4-x, & 2<x<4\end{cases}$$

*(GBTU, 2011)*

**Solution.** Here, we have $f(x)=\begin{cases}x, & 0<x<2\\ 4-x, & 2<x<4\end{cases}$ ...(1)

We have to find out half range Fourier sine series of (1).

$$b_n=\frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx \qquad (c=4)$$

$$=\frac{2}{4}\int_0^2 x\sin\frac{n\pi x}{4}\,dx+\frac{2}{4}\int_2^4(4-x)\sin\frac{n\pi x}{4}\,dx$$

$$=\frac{1}{2}\left[x\frac{4}{n\pi}\left(-\cos\frac{n\pi x}{4}\right)-(1)\frac{16}{n^2\pi^2}\left(-\sin\frac{n\pi x}{4}\right)\right]_0^2$$

$$+\frac{1}{2}\left[(4-x)\frac{4}{n\pi}\left(-\cos\frac{n\pi x}{4}\right)-(-1)\frac{16}{n^2\pi^2}\left(-\sin\frac{n\pi x}{4}\right)\right]_2^4$$

$$=\frac{1}{2}\left[(2)\frac{4}{n\pi x}\left(-\cos\frac{n\pi 2}{4}\right)-\frac{16}{n^2\pi^2}\left(-\sin\frac{n\pi 2}{4}\right)\right]$$

$$+\frac{1}{2}\left[0-\frac{16}{n^2\pi^2}\sin\frac{n\pi 4}{4}-(4-2)\frac{4}{n\pi}\left(-\cos\frac{n\pi}{2}\right)+\frac{16}{n^2\pi^2}\sin\frac{2n\pi}{4}\right]$$

$$=-\frac{4}{n\pi}\cos\frac{n\pi}{2}+\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}-\frac{16}{2n^2\pi^2}\sin n\pi+\frac{4}{n\pi}\cos\frac{n\pi}{2}+\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$b_1=\frac{16}{\pi^2},\quad b_2=0,\quad b_3=-\frac{16}{3^2\pi^2},\quad b_4=0,\quad b_5=-\frac{16}{5^2\pi^2}\ldots\ldots+$$

$$f(x)=b_1\sin\frac{\pi x}{4}+b_2\sin\frac{2\pi x}{4}+b_3\sin\frac{3\pi x}{4}+b_4\sin\frac{4\pi x}{4}+b_5\sin\frac{5\pi x}{4}$$

Hence $f(x)=\frac{16}{\pi^2}\left[\sin\frac{\pi x}{4}-\frac{1}{3^2}\sin\frac{3\pi x}{4}+\frac{1}{5^2}\sin\frac{5\pi x}{4}-\ldots\ldots\right]$ **Ans.**

**Example 28.** *Prove that*

$$f(x)=\frac{l}{2}-x=\frac{l}{\pi}\sum_1^{\infty}\frac{1}{n}\sin\frac{2n\pi x}{l},\ 0<x<l$$

**Solution.** $f(x)=\frac{1}{2}-x$

$$a_0=\frac{1}{l/2}\int_0^l f(x)\,dx=\frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)dx\,\frac{2}{l}\left[\frac{lx}{2}-\frac{x^2}{2}\right]_0^l=0$$

$$a_n=\frac{1}{l/2}\int_0^l f(x)\cos\frac{n\pi x}{l/2}\,dx=\frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\cos\frac{2n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\left(\frac{l}{2}-x\right)\frac{l}{2n\pi}\sin\frac{2n\pi x}{l}+(-1)\frac{l^2}{4n^2\pi^2}\cos\frac{2n\pi x}{l}\right]_0^l=\frac{2}{l}\left[0-\frac{l^2}{4n^2\pi^2}\cos 2n\pi+\frac{l^2}{4n^2\pi^2}\right]$$

$$=\frac{2}{l}\frac{l^2}{4n^2\pi^2}(-\cos 2n\pi+1)=\frac{l}{2n^2\pi^2}(-1+1)=0$$

$$b_n=\frac{1}{l/2}\int_0^l f(x)\sin\frac{n\pi x}{l/2}\,dx=\frac{2}{l}\int_0^l\left(\frac{1}{2}-x\right)\sin\frac{2n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\left(\frac{1}{2}-x\right)\left(-\frac{1}{2n\pi}\cos\frac{2n\pi x}{l}\right)-(-1)\left(-\frac{l^2}{4n^2\pi^2}\sin\frac{2n\pi x}{l}\right)\right]_0^l$$

$$=\frac{2}{l}\left[\frac{l}{2}\frac{l}{2n\pi}\cos 2n\pi-0+\frac{l}{2}\cdot\frac{l}{2n\pi}(1)\right]=\frac{2}{l}\left[\frac{l^2}{2n\pi}\right]=\frac{l}{n\pi}$$

Fourier series is

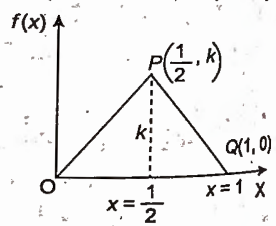
$$f(x)=\frac{a_0}{2}+a_1\cos\frac{n\pi x}{l/2}+a_2\cos\frac{2n\pi x}{l/2}+a_3\cos\frac{3n\pi x}{l/2}+\ldots$$

$$+b_1\sin\frac{n\pi x}{l/2}+b_2\sin\frac{2n\pi x}{l/2}+b_3\sin\frac{3n\pi x}{l/2}+\ldots$$

$$\frac{l}{2}-x=\frac{l}{\pi}\sin\frac{2\pi}{l}+\frac{l}{2\pi}\sin\frac{4\pi x}{l}=\frac{l}{3\pi}\sin\frac{6\pi x}{l}+\ldots=\frac{l}{\pi}\sum_1^\infty\frac{1}{n}\sin\frac{2n\pi x}{l}$$ **Proved.**

**Example 29.** *Find the half period sine series for f(x) given in the range (0, l) by the graph OPQ as shown in figure.* *(U.P. II semester, 2009)*

**Solution.** The equation of line $OP$ is $y=\frac{kx}{\frac{l}{2}}\Rightarrow y=\frac{2kx}{l}$




and the equation of the line $PQ$ is $y=-\frac{kx}{\frac{l}{2}}\Rightarrow y=-\frac{2kx}{l}$

$f(x)$ is the half period

$$f(x)=\begin{cases}\frac{2kx}{l}, & 0<x<\frac{l}{2}\\ -\frac{2kx}{l}+2k, & \frac{l}{2}<x<l\end{cases}$$



$f(x)$ is half period series. It is to be expended as sine series.
Here, $a_0=0$ and $a_n=0$

$$b_n=\frac{2}{l}\int_0^{\frac{l}{2}}f(x)\sin\frac{n\pi x}{l}\,dx+\frac{2}{l}\int_{\frac{l}{2}}^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^{\frac{l}{2}}\frac{2kx}{l}\sin\frac{n\pi x}{l}\,dx+\frac{2}{l}\int_{\frac{l}{2}}^l\left(\frac{-2kx}{l}+2k\right)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{4k}{l^2}\int_0^{\frac{l}{2}}x\sin\frac{n\pi x}{l}\,dx+\frac{4k}{l^2}\int_{\frac{l}{2}}^l(-x+l)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{4k}{l^2}\left[x\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{\frac{l}{2}}$$

$$+\frac{4k}{l^2}\left[(-x-l)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right)-(-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{\frac{l}{2}}^l$$

$$=\frac{4k}{l^2}\left[-\frac{l}{2}\left(\frac{l}{n\pi}\right)\cos\frac{n\pi\frac{l}{2}}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi\frac{l}{2}}{l}\right]$$

$$+\frac{4k}{l^2}\left[(-l+l)\left(-\frac{l}{n\pi}\cos n\pi\right)-\frac{l^2}{n^2\pi^2}\sin n\pi-\left(-\frac{l}{2}+l\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi\frac{l}{2}}{l}\right)+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi\frac{l}{2}}{l}\right]$$

$$=\frac{4k}{l^2}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]+\frac{4k}{l^2}\left[0-0+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{4k}{l^2}\left(\frac{l^2}{2n\pi}\right)\left[-\cos\frac{n\pi}{2}+\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]+\frac{4k}{l^2}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{4k}{l^2}\left(\frac{l^2}{2n\pi}\right)\left[-\cos\frac{n\pi}{2}+\frac{2}{n\pi}\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}+\frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$=\frac{2k}{n\pi}\left[\frac{4}{n\pi}\sin\frac{n\pi}{2}\right]=\frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Hence, Fourier series of $f(x)$ is

$$f(x)=\frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$ **Ans.**

## EXERCISE 10.4

1. Find the Fourier series to represent $f(x)$, where

$$f(x)=\begin{cases}-a, & -c<x<0\\ a, & 0<x<c\end{cases}$$

**Ans.** $\frac{4a}{\pi}\left[\sin\frac{\pi x}{c}+\frac{1}{3}\sin\frac{3\pi x}{c}+\frac{1}{5}\sin\frac{5\pi x}{c}+\ldots.\right]$

2. Find the half-range sine series for the function

$f(x)=2x-1\quad 0<x<1$

**Ans.** $-\frac{2}{\pi}\left[\sin 2\pi x+\frac{1}{2}\sin 4\pi x+\frac{1}{3}\sin 6\pi x+\ldots\right]$

3. Express $f(x) = x$ as a cosine, half range series in $0 < x < 2$.

**Ans.** $1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + ...\right]$

4. Find the Fourier series of the function

$$f(x) = \begin{cases} -2 & \text{for } -4 < x < -2 \\ x & \text{for } -2 < x < 2 \\ 2 & \text{for } 2 < x < 4 \end{cases}$$

**Ans.** $\frac{4}{\pi} + \frac{8}{\pi^2}\sin\frac{\pi x}{4} - \frac{2}{\pi}\sin\frac{2\pi x}{4} + \left(\frac{4}{3\pi} - \frac{8}{3^2\pi}\right)\sin\frac{3\pi x}{4} - \frac{2}{2\pi}\sin\frac{4\pi x}{4} + ...$

5. Find the Fourier series to represent $f(x) = x^2 - 2$ from $-2 < x < 2$.

**Ans.** $-\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2^2} - \frac{1}{4}\cos\pi x + \frac{1}{9}\cos\frac{3\pi x}{2} + ...\right]$

6. If $f(x) = e^x$, $-c < x < c$, show that

$$f(x) = (e^c - e^{-c})\left\{\frac{1}{2c} - c\left(\frac{1}{c^2+\pi^2}\cos\frac{\pi x}{c} - \frac{1}{c^2+4\pi^2}\cos\frac{2\pi x}{c} + ...\right) - \pi\left(\frac{1}{c^2+\pi^2}\sin\frac{\pi x}{c} - \frac{2}{c^2+4\pi^2}\sin\frac{2\pi x}{c} ...\right)\right\}$$

7. A sinusoidal voltage E sin $\omega t$ is passed through a half wave rectifier which clips the negative portion of the wave. Develop the resulting portion of the function

$$u(t) = \begin{cases} 0, & \text{when } -\frac{T}{2} < t < 0 \\ E\sin\omega t, & \text{when } 0 < t < \frac{T}{2} \end{cases} \qquad \left(T = \frac{2\pi}{\omega}\right)$$

**Ans.** $\frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left[\frac{1}{1.3}\cos 2\omega t + \frac{1}{3.5}\cos 4\omega t + \frac{1}{5.7}\cos 6\omega t + ...\right]$

8. A periodic square wave has a period 4. The function generation the square is

$$f(t) = \begin{cases} 0 & \text{for } -2 < t < -1 \\ k & \text{for } -1 < t < 1 \\ 0 & \text{for } 1 < t < 2 \end{cases}$$

Find the Fourier series of the function. **Ans.** $f(t) = \frac{k}{2} + \frac{2k}{\pi}\left[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + ...\right]$

9. Find a Fourier series to represent $x^2$ in the interval $(-l, l)$

**Ans.** $\frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\cos\pi x - \frac{\cos\pi x}{2^2} + \frac{\cos 3\pi x}{3^2}\right]$

## 10.15 PARSEVAL'S FORMULA

$$\int_{-c}^{c}[f(x)]^2 dx = c\left\{\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right\}$$

We know that $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{c} + b_n\sin\frac{n\pi x}{c}\right)$ ... (1)



Multiplying (1) by $f(x)$, we get

$$[f(x)]^2 = \frac{a_0}{2}f(x) + \sum_{n=1}^{\infty}a_n f(x)\cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty}b_n f(x)\sin\frac{n\pi x}{c} \quad ... (2)$$

Inter grating term by term from $-c$ to $c$, we have

$$\int_{-c}^{c}[f(x)]^2 dx = \frac{a_0}{2}\int_{-c}^{c}f(x)dx + \sum_{n=1}^{\infty}a_n\int_{-c}^{c}f(x)\cos\frac{n\pi x}{c}dx + \sum_{n=1}^{\infty}b_n\int_{-c}^{c}f(x)\sin\frac{n\pi x}{c}dx \quad ... (3)$$

In article 10.11, we have the following results

$$\int_{-c}^{c}f(x)dx = ca_0$$

$$\int_{-c}^{c}f(x)\cos\frac{n\pi x}{c}dx = ca_n$$

$$\int_{-c}^{c}f(x)\sin\frac{n\pi x}{c}dx = cb_n$$

On putting these integral in (3), we get

$$\int_{-c}^{c}[f(x)]^2 dx = c\frac{a_0^2}{2} + \sum_{n=1}^{\infty}ca_n^2 + \sum_{n=1}^{\infty}cb_n^2 = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

This is the Parseval's formula

**Note.**

1. If $0 < x < 2c$, then $\int_0^{2c}[f(x)]^2 dx = c\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$

2. If $0 < x < c$ (Half range sine series), $\int_0^c[f(x)]^2 dx = \frac{c}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}a_n^2\right]$

3. If $0 < x < c$ (Half range sine series), $\int_0^c[f(x)]^2 dx = \frac{c}{2}\left[\sum_{n=1}^{\infty}b_n^2\right]$

4. R.M.S. $= \left\{\frac{\int_a^b[f(x)]^2 dx}{b-a}\right\}^{\frac{1}{2}}$

**Example 30.** *By using the series for $f(x) = 1$ in $0 < x < \pi$ show that*

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

**Solution.** Sine series is $f(x) = \Sigma b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = \frac{-2}{n\pi}[\cos n\pi - 1] = \frac{-2}{n\pi}[(-1)^n - 1]$$

$$= \frac{4}{n\pi} \text{ if } n \text{ is odd}$$
$$= 0 \quad \text{if } n \text{ is even}$$

Then the sine series is

$$1 = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \ldots$$

$$\int_0^c [f(x)]^2 dx = \frac{c}{2}[b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + \ldots]$$

$$\int_0^\pi (1)^2 dx = \frac{\pi}{2}\left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \left(\frac{4}{7\pi}\right)^2 + \ldots\right]$$

$$[x]_0^\pi = \left(\frac{\pi}{2}\right)\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right]$$

$$\pi = \frac{\pi}{2}\left(\frac{16}{\pi^2}\right)\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

**Proved.**

**Example 31.** *If* $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$ *using half range cosine series, show that*

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$$

**Solution.** Half range cosine series is

$$f(x) = \frac{a_0}{2} + \Sigma a_n \cos\frac{n\pi x}{c}$$

where $a_0 = \frac{2}{c}\int_0^c f(x)\,dx = \frac{2}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$

$$= \pi\left(\frac{x^2}{2}\right)_0^1 + \pi\left(2x - \frac{x^2}{2}\right)_1^2 = \frac{\pi}{2} + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = \pi$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{2}\left[\int_0^1 \pi x\cos\frac{n\pi x}{2}\,dx + \int_1^2 \pi(2-x)\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \pi\left[x\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_0^1 + \pi\left[(2-x)\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}} - (-1)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{4}}\right)\right]_1^2$$

$$= \pi\left[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\right] + \pi\left[0 - \frac{4}{n^2\pi^2}\cos n\pi - \frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$



$$= \pi\left[\frac{8}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2} - \frac{4}{n^2\pi^2}\cos n\pi\right] = \frac{4}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - \cos n\pi\right]$$

$$a_1 = 0,\ a_2 = \frac{-4}{\pi},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{-4}{9\pi}\ldots$$

$$\int_0^c [f(x)]^2 dx = \frac{c}{2}\left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

$$\int_0^1 (\pi x)^2 dx + \int_1^2 \pi^2(2-x)^2 dx = \frac{2}{2}\left[\frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots\right]$$

$$\pi^2\left[\frac{x^3}{3}\right]_0^1 - \pi^2\left[\frac{(2-x)^3}{3}\right]_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \ldots$$

$$\frac{\pi^2}{3} - \pi^2\left(0 - \frac{1}{3}\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2}\left[1 + \frac{1}{81} + \ldots\right]$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^2}{6} = \frac{16}{\pi^2}\left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

**Ans.**

**Example 32.** *Prove that for* $0 < x < \pi$

(*a*) $x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \ldots\right]$

(*b*) $x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \ldots\right]$

*Deduce from (a) and (b) respectively that*

(*c*) $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$ (*d*) $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^4}{945}$

**Solution.** (*a*) Half range consine series

$$a_0 = \frac{2}{\pi}\int_0^\pi x(\pi - x) = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[0 - \frac{\pi(-1)^n}{n^2} + 0 - \frac{\pi}{n^2}\right] = \frac{2}{\pi}\left(\frac{\pi}{n^2}\right)[-(-1)^n - 1]$$

$$= -\frac{4}{n^2} \quad \text{(when } n \text{ is even)}$$

$$= 0 \quad \text{(when } n \text{ is odd)}$$

Hence,
$$x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \ldots\right]$$ **Proved.**

By Parseval's formula
$$\frac{2}{\pi}\int_0^\pi x^2(\pi - x)^2\,dx = \frac{a_0^2}{2} + \Sigma a_n^2$$

$$\frac{2}{\pi}\int_0^\pi (\pi^2x^2 - 2\pi x^3 + x^4)\,dx = \frac{1}{2}\left(\frac{\pi^4}{9}\right) + 16\left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^2x^3}{3} - \frac{2\pi x^4}{4} + \frac{x^5}{5}\right]_0^\pi = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right]$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{3} - \frac{2\pi^5}{4} + \frac{\pi^5}{5}\right] = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right]$$

$$\frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=1}^{\infty}\frac{1}{n^4} \quad \text{or} \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

*(b) Half range sine series*

$$b_n = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[-2\frac{(-1)^n}{n^3} + \frac{2}{n^3}\right] = \frac{4}{\pi n^3}[-(-1)^n + 1]$$

$$= \frac{8}{n^3\pi} \quad \text{(when } n \text{ is odd)}$$

$$= 0 \quad \text{(where } n \text{ is even)}$$

$$\therefore \quad x(\pi - x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right]$$ **Proved.**

By Parseval's formula

$$\frac{2}{\pi}\int_0^\pi x^2(\pi - x^2)\,dx = \Sigma b_n^2$$

$$\frac{\pi^2}{15} = \frac{64}{\pi^2}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\frac{\pi^4}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots$$

Let $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right)$



$$S = \frac{\pi^4}{960} + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right) = \frac{\pi^4}{960} + \frac{1}{2^6}\left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots\right]$$

$$S = \frac{\pi^4}{960} + \frac{S}{64}$$

$$S - \frac{S}{64} = \frac{\pi^4}{960} \Rightarrow \frac{63}{64}S = \frac{\pi^4}{960}$$

$$S = \frac{\pi^4}{960} \times \frac{64}{63} = \frac{\pi^4}{945}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^4}{945}$$ **Proved.**

## EXERCISE 10.5

1. Prove that in $0 < x < c$,

$$x = \frac{c}{2} - \frac{4c}{\pi^2}\left(\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \frac{1}{5^2}\cos\frac{5\pi x}{c} + \ldots\right)$$ and deduce that

(i) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$ (ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$

## 10.16 FOURIER SERIES IN COMPLEX FORM

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + a_2\cos\frac{2\pi x}{l} + \ldots + a_n\cos\frac{n\pi x}{l} + \ldots$$
$$+ b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + \ldots + b_n\sin\frac{n\pi x}{l} + \ldots \quad \ldots (1)$$

We know that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ and $x = \frac{e^{ix} - e^{-ix}}{2i}$

On putting the values of $\cos x$ and $\sin x$ in (1), we get

$$f(x) = \frac{a_0}{2} + a_1\frac{e^{\frac{i\pi x}{l}} + e^{\frac{-i\pi x}{l}}}{2} + a_2\frac{e^{\frac{2i\pi x}{l}} + e^{\frac{-2i\pi x}{l}}}{2} + \ldots + b_1\frac{e^{\frac{i\pi x}{l}} - e^{\frac{i\pi x}{l}}}{2i} + b_2\frac{e^{\frac{2i\pi x}{l}} - e^{\frac{-2i\pi x}{l}}}{2i} + \ldots$$

$$= \frac{a_0}{2} + (a_1 - ib_1)e^{\frac{i\pi x}{l}} + (a_2 - ib_2)e^{\frac{2i\pi x}{l}} + \ldots + (a_1 + ib_1)e^{\frac{-i\pi x}{l}} + (a_2 + ib_2)e^{\frac{-2i\pi x}{l}} + \ldots$$

$$= c_0 + c_1 e^{\frac{i\pi x}{l}} + c_2 e^{\frac{2i\pi x}{l}} + \ldots + c_{-1} e^{\frac{-i\pi x}{l}} + c_{-2} e^{\frac{-2i\pi x}{l}} + \ldots$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{\frac{in\pi x}{l}} + \sum_{n=1}^{\infty} c_{-n} e^{\frac{-in\pi x}{l}}$$

$$c_n = \frac{1}{2}(a_n - ib_n),\; c_{-n} = \frac{1}{2}(a_n - ib_n)$$

where $c_0 = \frac{a_0}{2} = \frac{1}{2}\frac{1}{l}\int_0^{2l} f(x)\,dx$